# Low-Rank Phase Retrieval with Structured Tensor Models

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## Problem and Challenges

**Problem Statement:**
Recover a sequence of $q$ signals (typically images) $\mathbf{x}_k \in \mathbb{C}^n$ given sampling matrices $\mathbf{A}_k \in \mathbb{C}^{n\times m}$ and measurements $\mathbf{y}_k \in \mathbb{R}^m$, where

$$\mathbf{y}_k = |\mathbf{A}_k^\mathsf{T}\mathbf{x}_k|, \quad k = 1, \dots, q.$$

**Existing Solutions:**

- AltMinTrunc [1] and AltMinLowRaP [2] recover a low-rank matrix $\mathbf{X}$ constructed by vectorizing and stacking each signal
- AltMinLowRaP has strong theoretical guarantees regarding sample complexity for accurate recovery
- Both fail to converge (or converge to a poor local minimum) when the number of measurements is more limited ($m \ll n$)

## Low-Rank Phase Retrieval (LRPR)

**Algorithm Overview:**

Take top $r$ eigenvectors $\sum_{k=1}^{q}\sum_{i=1}^{m} y_{i,k}^2 \mathbf{a}_{i,k}\mathbf{a}_{i,k}^\top \mathbf{1}_{\{y_{i,k}^2 \le \frac{C}{mq}\sum_{i,k} y_{i,k}^2\}}$

Initialize $\mathbf{U}$

Solve PR with $\mathbf{y}$, $\mathbf{U}^\top\mathbf{A}_k$

$\mathbf{b}_k \leftarrow \mathrm{RWF}(\mathbf{y}_k, \mathbf{U}^\top\mathbf{A}_k)$

Update $\mathbf{b}_k$

$\mathbf{x}_k = \mathbf{U}\mathbf{b}_k$

Update $\mathbf{x}_k$

$\mathbf{C}_k = \mathrm{Diag}(\mathrm{Phase}(\mathbf{A}_k^\top\mathbf{x}_k))$

Update Phases $\mathbf{C}_k$

$\mathbf{U} \leftarrow \underset{\mathbf{U}}{\mathrm{argmin}} \sum_{k=1}^{q} \|\mathbf{C}_k\mathbf{y}_k - \mathbf{A}_k^\top \underbrace{\mathbf{U}\mathbf{b}_k}_{\mathbf{x}_k}\|_2$

Update $\mathbf{U}$

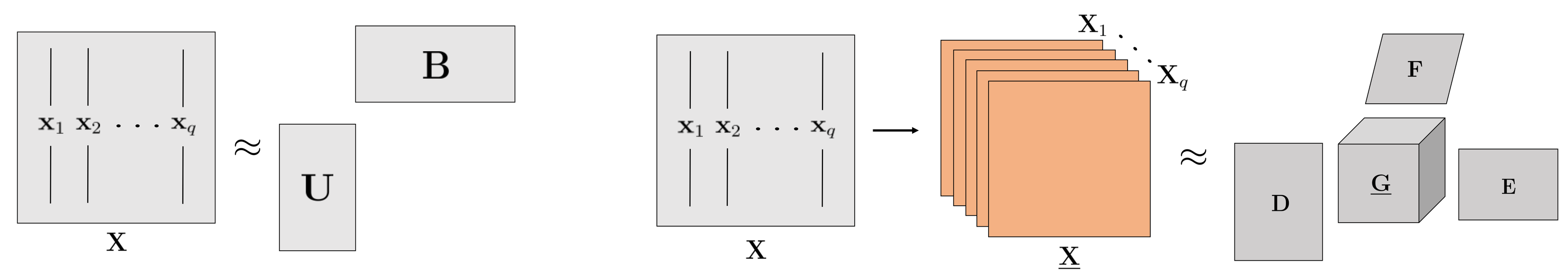
Repeat $T$ for $k = 1, \dots, q$

## References

[1] S. Nayer, N. Vaswani, and Y. C. Eldar, "Low rank phase retrieval", IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pp. 4446–4450, 2017

[2] S. Nayer, P. Narayanamurthy, and N. Vaswani, "Provable low rank phase retrieval", IEEE Transactions on Information Theory, vol. 66, no. 9, pp. 5875–5903, 2020

## Tucker-Structured Low-Rank Phase Retrieval (TSPR)




LRPR TSPR

**Main Ideas:**

- Reduce sample complexity by reducing the number of parameters by assuming that the data "lives" on a simpler manifold
- Un-vectorize each signal $\mathbf{x}_k$ and stack them to create a tensor which we factorize using the Tucker decomposition

**Algorithm Overview:**

Take top eigenvector of $\sum_{i=1}^{m} y_{i,k}^2 \mathbf{a}_{i,k}\mathbf{a}_{i,k}^\top \mathbf{1}_{\{y_{i,k}^2 \le \frac{C}{mq}\sum_{i,k} y_{i,k}^2\}}$ for $k = 1, \dots, q$

Initialize Tensor Slices → Reshape and Construct Tensor ($\underline{\mathbf{X}}$) → Initialize Factors with HOSVD ($\mathbf{D}$, $\underline{\mathbf{G}}$, $\mathbf{E}$, $\mathbf{F}$)

Solve PR with $\mathcal{M}_3(\underline{\mathbf{G}})(\mathbf{E}\otimes\mathbf{D})^\top\mathbf{a}_{i,k}$

$\mathbf{f}_k \leftarrow \mathrm{RWF}(\mathbf{y}_k, \mathcal{M}_3(\underline{\mathbf{G}})(\mathbf{E}\otimes\mathbf{D})^\top\mathbf{a}_{i,k})$

Update $\mathbf{f}_k$

$\mathbf{x}_k = (\mathbf{f}_k \otimes \mathbf{E} \otimes \mathbf{D})\mathrm{vec}(\underline{\mathbf{G}})$

Update $\mathbf{x}_k$

$\mathbf{C}_k = \mathrm{Diag}(\mathrm{Phase}(\mathbf{A}_k^\top\mathbf{x}_k))$

Update Phases $\mathbf{C}_k$

$\mathbf{D} \leftarrow \underset{\mathbf{D}}{\mathrm{argmin}} \sum_{k=1}^{q} \|\mathbf{C}_k\mathbf{y}_k - \mathbf{A}_k^\top(\mathbf{f}_k\otimes\mathbf{E}\otimes\mathbf{D})\mathrm{vec}(\underline{\mathbf{G}})\|_2$

Update $\mathbf{D}$

$\mathbf{E} \leftarrow \underset{\mathbf{E}}{\mathrm{argmin}} \sum_{k=1}^{q} \|\mathbf{C}_k\mathbf{y}_k - \mathbf{A}_k^\top(\mathbf{f}_k\otimes\mathbf{E}\otimes\mathbf{D})\mathrm{vec}(\underline{\mathbf{G}})\|_2$

Update $\mathbf{E}$

$\underline{\mathbf{G}} \leftarrow \underset{\underline{\mathbf{G}}}{\mathrm{argmin}} \sum_{k=1}^{q} \|\mathbf{C}_k\mathbf{y}_k - \mathbf{A}_k^\top(\mathbf{f}_k\otimes\mathbf{E}\otimes\mathbf{D})\mathrm{vec}(\underline{\mathbf{G}})\|_2$

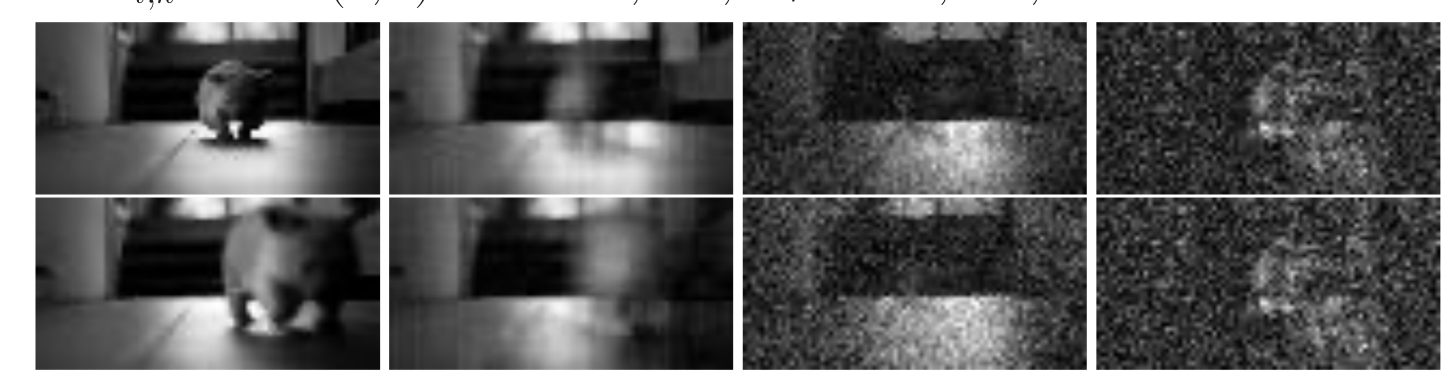
Update $\underline{\mathbf{G}}$

Repeat $T$ for $k = 1, \dots, q$

## Experimental Results

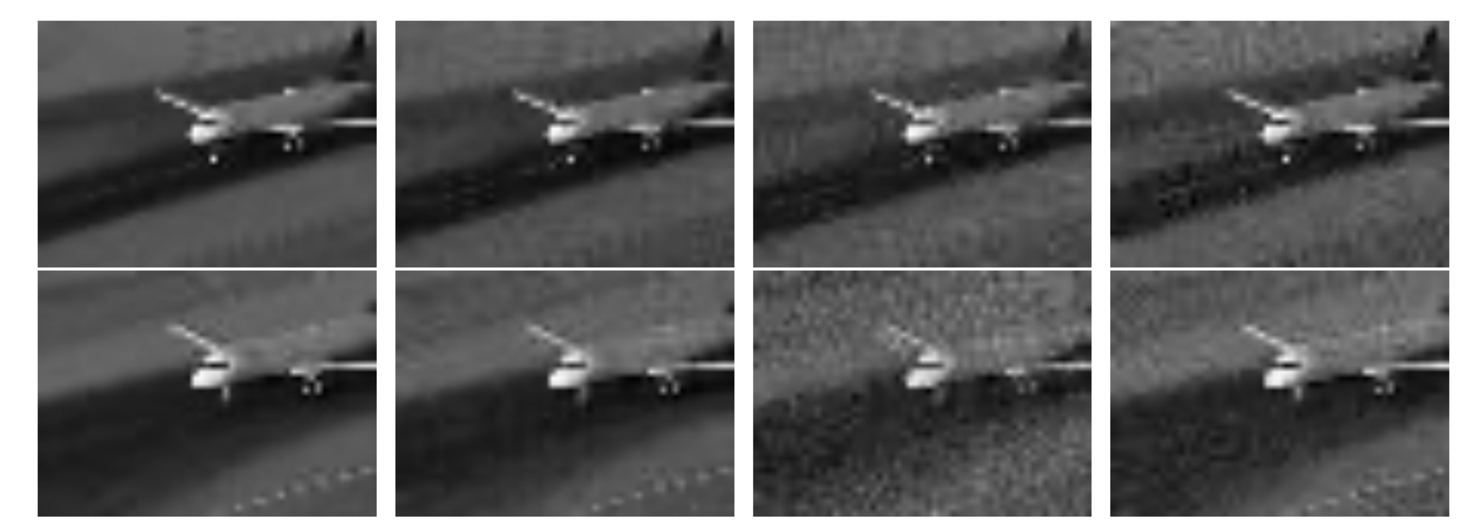
**Goal:** Observe the performance of TSPR compared to existing algorithms by recovering a grayscale video in both the under and over-determined regimes

**Under-Sampled Case** $(m \ll n)$**:** Recover mouse video $(40 \times 80 \times 90)$ from $\mathbf{a}_{i,k} \sim \mathcal{CN}(0, \mathbf{I})$ for $i = 1, \dots, m$, $k = 1, \dots, 90$ with $m = 0.75n$



Original TSPR AltMinLowRaP AltMinTrunc

**Over-Sampled Case** $(m \gg n)$**:** Recover plane video $(40 \times 55 \times 90)$ from Coded Diffraction Patterns (CDP) with $m = 2n$



Original TSPR AltMinLowRaP AltMinTrunc

**Remarks:**

- AltMinLowRaP converges to a poor local minimum when the number of measurements is more limited
- TSPR converges to a better solution in the under-sampled regime, but with a visual effect caused by the low-rank assumption on each of the images
- The effect highlights a tradeoff between the choice of the ranks and accuracy of the reconstruction
- TSPR performs comparably well (both visually and numerically) in the over-sampled regime

## Conclusion and Future Work

- Observed that using tensor models for LRPR empirically reduced the sample complexity needed for accurate recovery
- Continuing work involves developing theoretical results for the initialization and descent steps of TSPR