A Note on the Region of Convergence

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1 Region of Convergence

To start, you might be wondering why we even need the region of convergence (ROC), or why it's important to specify the ROC when talking about z-transforms. The main reason for this is because it turns out that there can be more than one signal with the same z-transform, and the way to differentiate them is to specify the ROC.

Now, how do we determine the ROC? Let's say that there was a pole at a for $X(z)$. We can determine the ROC by drawing a circle at radius $|a|$. This splits up the complex plane into a bunch of annular regions, and we can determine the ROC by those regions. The following are **important** facts about the ROC and how it's affected by the signals:

- The ROC contains no poles! This is why when we specify our ROC for our z-transforms, it's strictly less than or greater than the absolute value of the pole (i.e. $|z| > a$ or $|z| < a$).
- If our signal $x[n]$ is right-sided (causal) and we have a pole at $|z| = a$, then our ROC is also right-sided such that the ROC satisfies $|z| > a$.
- If our signal $x[n]$ is left-sided (anti-causal) and we have a pole at $|z| = a$, then our ROC is also left-sided such that the ROC satisfies $0 < |z| < a$.
- If our signal $x[n]$ is two-sided (neither causal or anti-causal), then the ROC is also two-sided and the ROC is a disc (or ring) defined by its poles.
- The transfer function $H(z)$ is **stable** if and only if the ROC contains the unit circle, and hence, its DTFT exists.
- A **casual system** is **stable** if and only if all of its poles are **inside** the unit circle.
- An **anti-causal system** is **stable** if and only if all of its poles are **outside** of the unit circle.

As a smaller note, "if and only if" is a biconditional statement, meaning that the statement can go either way. For example, if all of the poles of a system are inside the unit circle, then a causal system is stable.